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LETTER TO THE EDITOR

Static critical exponents from the dynamics of damage spreading and overhangs in the Ising model with temperature gradient

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Abstract. By studying damage spreading in spin models in a temperature gradient, it is possible to determine not only the critical temperature, and the correlation length exponent ν, but also a second critical exponent β with high precision. This is done by studying the dynamics of damage spreading, in particular the overhangs which develop in the damage front. We demonstrate the method for the two-dimensional Ising model using heat bath dynamics, and discuss differences with Metropolis dynamics.

The aim of this letter is to introduce a new numerical technique to determine critical exponents in spin models, based on damage spreading [1] in a temperature gradient [2] and overhang statistics [3, 4], first developed in connection with cellular automata. It is simple to implement, simple to analyse, and leads to precise results with only little computational effort.

We will use in this letter the two-dimensional Ising model as a testing ground for our method. The spin σ_i, associated with node i in a square lattice, takes on either of the two values ± 1. Interactions between the spins are described through the Hamiltonian $H = -\Sigma_{\langle ij\rangle} \sigma_i \sigma_j$ where $\langle ij\rangle$ indicates nearest neighbours. In order to model the behaviour of this system at a finite temperature T, we use heat bath dynamics [5]. For each node i, a random number $0 < r_i(t) < 1$ is generated. If $r_i(t) < p_i(t) = (1+\exp[-2\,\Sigma_{\langle ij\rangle}\, \sigma_j(t)/T])^{-1}$, the spin $\sigma_i(t+1)$ is set to $+1$, otherwise it is set to -1. t refers to Monte Carlo time.

The concept of 'damage' in the Ising model refers to the following construction [1]. Two lattices A and B, of equal size, are initialized in states $\{\sigma_i^A(0)\}$ and $\{\sigma_i^B(0)\}$. Both lattices are then subjected to the same dynamics, including using the same series of random numbers $r_i(t)$. The damage associated with node i is defined as $d_i(t) = \frac{1}{2}|\sigma_i^A(t) - \sigma_i^B(t)|$ which is 1 if the spins in the two lattices are different, and zero if they are equal. The total damage is defined as $D(t) = (1/N)\,\Sigma_i\, d_i(t)$, where N is the number of nodes in each of the two lattices A and B.

Coniglio *et al* [6] have demonstrated that with heat bath dynamics, the total damage $D(t)$ heals, i.e. goes to zero with increasing time, if the temperature T is larger than the critical temperature $T_c = 2.269\,185$, while it remains finite for $T < T_c$. This was achieved by showing that as $t \to \infty$ the probability for a site to be damaged is equal to the spontaneous magnetization M.

Following Boissin and Herrmann [2] we consider an $L \times L$ lattice. In the x direction we introduce periodic boundary conditions. In the y direction, we add an extra row for $y=0$ where all spins are fixed, i.e. they are not updated. Thus, the spins in the row $y=1$ behave as if interacting with an external magnetic field imposed at the $y=0$ boundary. The spins belonging to the upper row $y=L$ form a free boundary. We now introduce a linear temperature gradient along the y direction, i.e. spins belonging to row y are held at a temperature

$$T(y)=T_{\min}+\frac{T_{\max}-T_{\min}}{L-1}(y-1)=T_{\min}+g(y-1) \tag{1}$$

where the gradient $g=(T_{\max}-T_{\min})/(L-1)$. The minimum and maximum temperatures are chosen so that $T_{\min}<T_c<T_{\max}$. In our numerical experiments, we typically set $T_{\min}=0.5T_c$ and $T_{\max}=1.5T_c$. In order to study damage propagation in this system, we set up two identical lattices, A and B, with identical initial spin configurations, except that in lattice A we let the spins along row $y=0$ be $+1$ while those of lattice B are set to -1.

As these two lattices develop in time, damage will propagate into the lattice from the 'permanently damaged' row $y=0$ ($T=T_{\min}$), eventually to settle into a steady state in the region where $T>T_c$. Since the damage front covers the critical region, its fluctuations as a function of time are critical and will, therefore, yield critical properties.

Boissin and Herrmann [6] (BH) studied the behaviour of the damage in this region by identifying the cluster of damaged spins that are connected to the permanently damaged spins at $y=0$, called the 'infinite cluster'. This was done by invoking the burning algorithm [7] at each Monte Carlo time step. Two damaged nodes belong to the same cluster if there exists a continuous path between them touching only damaged nodes. The infinite cluster has an outer boundary against the nodes that are connected to the row $y=L$ through continuous paths touching only *undamaged* nodes; see figure 1. Each node k belonging to this outer boundary, called a *front*, has a well defined temperature $T_k(t)$ associated with it. BH then defined an average temperature $T_{\mathrm{BH}}=(1/L)\sum_k\langle T_k(t)\rangle$ and a width $w_{\mathrm{BH}}=\sqrt{(1/L)\sum_k\langle(T_k(t)-T_{\mathrm{BH}})^2\rangle}$. Here $\langle\ldots\rangle$ is an average over time. As the gradient $g\to 0$, $T_{\mathrm{BH}}\to T_c$ and $w_{\mathrm{BH}}=g^{1-b}$. The exponent b was measured to be 0.51 ± 0.01. This value is consistent with

$$b=\frac{\nu}{1+\nu} \tag{2}$$

where $\nu=1$, is the correlation length critical exponent for the 2D Ising model. This exponent relation follows if the width of the front is proportional to the correlation length at T_{BH} [8].

Our method deviates from the method of BH first and foremost in that we do *not* identify an 'infinite' cluster of connected damaged sites. The spreading of damage in a d-dimensional spin system may be viewed as a complex directed growth process in $(d+1)$ dimensions, where the extra dimension is time. That is, in a spacetime diagram the damage shows a *connected* tree-like structure where the branches always stretch in the positive time direction. With such a picture in mind, it is not entirely clear what is the physical significance of *spatially* connected clusters at a given time slice t.

In [4], $(2+1)$-dimensional directed percolation was analysed. Even though this problem is quantitatively different from the Ising problem at hand, qualitatively they are very similar: the connected 'damage-tree' of the spin problem is analogous to the percolating cluster in the directed percolation problem. Thus, we analyse the Ising model in the same way as was done for the directed percolation problem in [4].

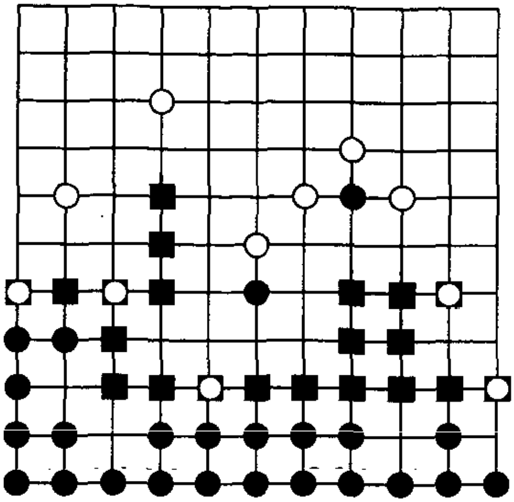

Figure 1. The damage configuration $d_i(t)$ at some time t. The row at the bottom of the figure is permanently damaged, and there is a temperature gradient from the bottom to the top of the figure. The damaged nodes are marked with either a filled circle, an open circle or a filled square. The nodes belonging to the damage front as defined in this paper, are marked by open circles, while those belonging to that defined by Boissin and Herrmann [2] are marked by the filled squares. Some nodes belong to both fronts, as is evident from the superposition of the two symbols. Note that the bottom boundary is at a temperature $T_{\min}$, and the top is at $T_{\max}$.

Rather than define the front through identifying the infinite spatial cluster, we define the position of the front at the coordinate x (measured parallel to the damaged line) as node with the largest y coordinate which is damaged; see figure 1. Thus, the position of the front defined this way becomes a single-valued function, $y = y(x, t; g)$, which is also a function of the temperature gradient g. The temperature corresponding to $y(x, t; g)$ is defined by $T(x, t; g) = T_{\min} + g[y(x, t; g) - 1]$. The average position of this front, measured in terms of temperature is

$$T_{\mathrm{eff}}(g) = \frac{1}{L} \sum_{x=1}^{L} \langle T(x, t; g) \rangle. \tag{3}$$

We also define the front width as

$$w(g) = \left(\frac{1}{L} \sum_{x=1}^{L} \langle (T(x, t; g) - T_{\mathrm{eff}}(g))^2 \rangle \right)^{1/2}. \tag{4}$$

In figure 2 we show T_{eff} as a function of $\sqrt{g}$, and in figure 3 a log–log plot of $w(g)$ as a function of $1/g$. The data are based on lattices ranging from $L = 30$ to 200, each with 10^5 Monte Carlo time steps. The T_{eff} extrapolates to $T_c = 2.27 \pm 0.01$, whereas the correct value is $T_c = 2.269\,185$. Figure 3 shows that $w(g) \sim g^{1-b}$, with $b = 0.48 \pm 0.01$. This value agrees with the exact value, $b = \frac{1}{2}$, obtained from equation (2). Thus, we conclude that the definition of the damage front presented here is a sensible one.

In [3, 4] the concept of an *overhang* in connection with directed processes was introduced. In terms of the Ising model, an overhang $j(x, t; g)$ is defined as the difference

$$j(x, t; g) = y(x, t; g) - y(x, t-1; g). \tag{5}$$

That is, it is the difference in the position of the front between two consecutive Monte Carlo time steps. In [4] it was argued that measuring the statistics of such overhangs

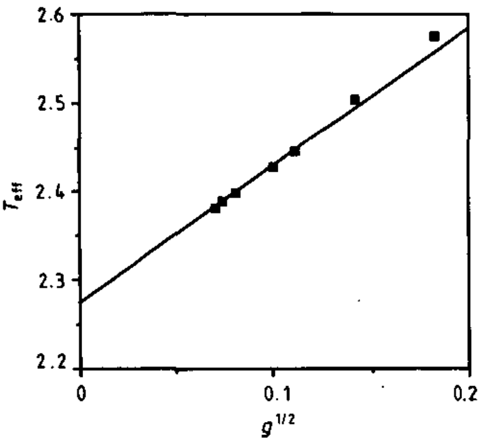

Figure 2. The average position of the front measured in terms of temperature plotted against the square root of the temperature gradient g. The linear fit—represented by the straight line—extrapolates to $T_c = 2.27$ for $g \to 0$.

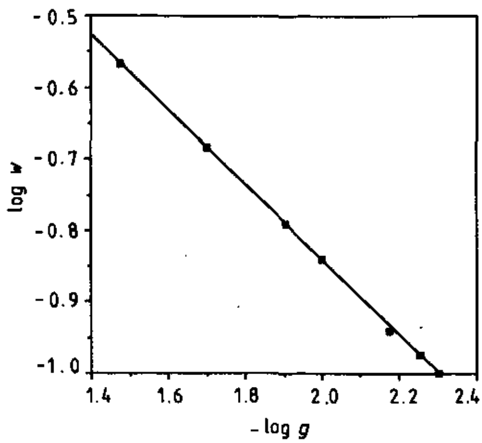

Figure 3. The width of the front measured in terms of temperature as a function of the inverse of the temperature gradient g. The slope of the straight line is 0.52 ± 0.01.

measures the fractal structure of the directed process. This fractal structure is in turn connected with the order parameter exponent β and the correlation length exponent ν. In particular, it was argued that the probability per row and per unit time, $N_{\pm}(j, g)$, to find an overhang of size j, in a $(d+1)$-dimensional directed process, is given by the scaling function

$$N_{\pm}(j, g) = j^{-a} n_{\pm}(jg^{b}) \tag{6}$$

where the subscript $\pm$ refers to positive or negative overhangs, and the crossover function $n_{\pm}$ approaches analytically a constant for small arguments, and going to zero faster than any power law for large arguments. The exponent b is that of equation (2), and a is given by

$$a = 4 - d - \beta/\nu \tag{7}$$

as long as the ratio $\beta/\nu < 1$. In the two-dimensional Ising model, $\beta = \frac{1}{8}$, so that $a = \frac{15}{8}$.

By measuring the overhang distribution, the exponents a and b may be determined, and subsequently ν and β. In practice, the best way to measure $N_{\pm}(j, g)$ is by measuring its *moments*. The kth moment of the positive overhangs is given by

$$\langle j^{k}(g)\rangle_{+} = \sum_{j=0}^{\infty} j^{k} N_{+}(j, g) = g^{b(a-k-1)} \sum_{j=0}^{\infty} j^{k} n_{+}(j) = A_{k} g^{-y(k)} \tag{8}$$

where

$$y(k) = b(k+1-a) = \frac{7}{16} + \frac{k}{2} \tag{9}$$

using the values of a and b for the two-dimensional Ising model. In figure 4 we show the second, fourth and sixth moments of the positive overhang distribution, measured for lattices of size ranging from $L = 10$ to $L = 200$, each with 2×10^{5} Monte Carlo updates, a run requiring approximately two days on a medium-sized workstation. It

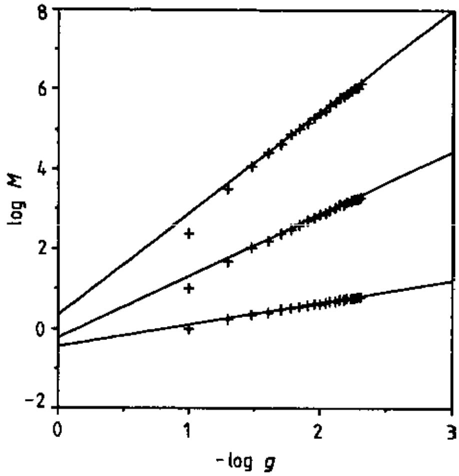

Figure 4. The second, fourth and sixth moments of the positive overhang distribution, $N_+(j, g)$, plotted as a function of $1/g$. The slopes of the best fits are respectively $y(2) = 0.55 \pm 0.01$, $y(4) = 1.56 \pm 0.01$, and $y(6) = 2.55 \pm 0.01$.

was also easy to write a parallel code that ran very efficiently on the Connection Machine CM2 and CM5. We determined the exponent $y(k)$ for all values of k between two and six, by least-squares fits as those shown in figure 5. In particular, we found $y(2) = 0.55 \pm 0.01$, $y(3) = 1.05 \pm 0.01$, $y(4) = 1.56 \pm 0.01$, $y(5) = 2.05 \pm 0.01$, and $y(6) = 2.55 \pm 0.01$. A best fit of these exponents is $y(k) = -0.45 + 0.50k$, which should be compared with equation (9), predicting $y(k) = -0.437 + 0.500k$. Using equations (9), (7) and (2) leads to our numerical determination $\nu = 1.00$ and $\beta = 0.10$, in excellent agreement with the exact values $\nu = 1$ and $\beta = 0.125$.

What happens if one uses Metropolis dynamics instead of heat bath? A major difference between these two is that unlike heat bath, in Metropolis the damage does not heal for $T > T_c$. It is, therefore, possible to have the damaged wall on the hot boundary, and the damage front propagating from the hot side. Doing that, and

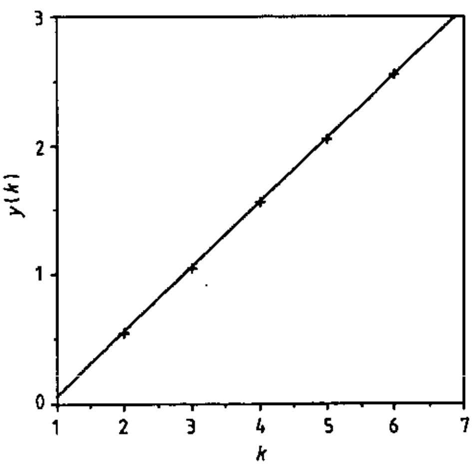

Figure 5. $y(k)$ as a function of k. The straight line is a best fit $y(k) = -0.45 + 0.50k$.

performing the simulations and data analysis the same way as for heat bath we obtain $\nu = \frac{1}{2}$: the *meanfield* value. Furthermore, equation (7) fails to give a non-zero value for β because it is based on hyperscaling which is violated by mean field. The reason for the mean-field results becomes apparent if we look at the average location of the hot-side damage front. This front is stopped in the hot side, short of the critical region by the big magnetization domains that form in the cold regions. Since the front is close to, but not in, the critical region it sees the mean-field properties of the phase transition. We, next, tried the simulation with the damage source on the cold wall, and the damage front propagating from the cold side, like the heat bath case. What happened here is the following. The damage front propagated from the cold side *along domain walls* and settled into an equilibrium position that covered the critical region, suggesting that fluctuations will be critical. However, damage *leaked* into the hot side, where it spread and formed hot-side damage clusters with their mean-field behaviour. The hot-side and cold-side clusters interacted, and the critical exponents we obtained seemed to be averages of the mean-field and true exponents. In short, Metropolis dynamics is much harder to use in this application than heat bath.

To conclude, we introduced a numerical method for determining the critical exponents ν and β in addition to the critical temperature in spin models. Unlike previous methods, which could determine ν and T_c but not β, our method is based on the *dynamics* of damage spreading and the overhangs that develop in the damage front in the presence of a temperature gradient. The method is very simple, and leads to accurate results with little computational effort. Finally, we mention that since the behaviour of the damage spreading is qualitatively the same as that of directed percolation, it should be possible to measure the dynamic critical exponent, z, by imposing a temperature gradient along the *time* direction [9].

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